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Stability of Interfaces in a Random Environment. A Rigorous Renormalization Group Analysis of a Hierarchical Model

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We study a hierarchical model of domain walls in a D-dimensional bond disordered Ising model at low temperatures. Using a renormalization group method inspired by the work of Bricmont and Kupiainen for the random field Ising model, we prove the existence of rigid interfaces at low enough temperatures in dimensions $D > 3$.

KEY WORDS: Disordered systems; interfaces; dilute Ising model; renormalization group; hierarchical model.

1. INTRODUCTION

The present paper is devoted to a study of the stability of interfaces in random media. Such problems may arise in a multitude of contexts, a prime example being the question of the existence of states describing the existence of domain walls in a dilute Ising model. This model is defined by the Hamiltonian

$$H_J = -\sum_{\langle ij \rangle} J_{ij} \sigma_i \sigma_j \tag{1.1}$$

where the sum is over all pairs of nearest neighbors of the lattice $\mathbb{Z}^d$, the σ_i are spin variables on the sites of this lattice, taking values ± 1, and the couplings J_{ij} are random variables in some probability space, typically chosen as independent and identically distributed. If the support of their distribution is contained in the positive real line, the Hamiltonian describes a ferromagnet for all possible realizations of the couplings. In this situation,

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it is easy to show that at sufficiently low temperatures there exist exactly two translational-invariant Gibbs states, describing the system with positive and negative magnetization, respectively, if the dimension of the lattice is at least two.

In the ordered case, i.e., where all J_{ij} take a constant positive value J, Dobrushin[7] has shown that in dimension $D \geqslant 3$, and for small enough temperatures, there exist also non-translational-invariant Gibbs states, in which two regions of positive and negative magnetization are separated by a domain wall. Such states can be obtained by applying + boundary conditions on the upper half of a box Λ and − boundary conditions on the lower half. In finite volume, this ensures that the resulting Gibbs state is characterized by the presence of an interface separating + and − spins. The question whether the infinite-volume Gibbs state obtained by letting the volume of the box Λ tend to infinity is a pure state describing a domain wall is tied to the question whether this interface will be "rigid" (i.e., stay localized in a finite region near the equatorial plane) or will undergo unbounded fluctuations as the volume of Λ increases. The latter has been shown to be the case in $d = 2$ by Gallavotti.[9]

The same question may of course be posed in the disordered case. The main new difficulty that enters here is the fact that while previously the "flat" surface was clearly the one minimizing the energy, this is generally no longer the case. Namely, the energy cost for having two neighboring spins of opposite value is now a space-dependent quantity, and the energy of an interface no longer depends solely on its surface area, but also explicitly on its position. Due to these added complexities, no rigorous results on the existence or nonexistence of Dobrushin states in this system are available.

On the heuristic level, this problem has received a quite considerable amount of attention over the last years. In most of these studies, rather than regarding the full model (1.1), a more simplified model for an interface in the presence of bond randomness was introduced. It consists of considering a surface S with fixed boundary ∂S chosen to be the equator of a D-dimensional box. The SOS (solid-on-solid) approximation consists in discarding all surfaces with "overhangs," the advantage of this approximation being that the remaining surfaces can be described as graphs of a function from the equatorial plane to the integers. In other words, an SOS surface is fully described by giving its "height" above any given point of a $(d = D - 1)$-dimensional lattice. The energy associated to such a surface is given by the sum of all the J_{ij} such that the bond $\langle ij \rangle$ transpierces the surface. A further simplification conventionally used is to set all J_{ij} corresponding to the parts of the surface perpendicular to the equatorial plane equal to a constant (e.g., 1), and to retain randomness only for those parallel to this plane.

This model has been studied extensively in dimension $D=2$, frequently under the name of "directed polymers." In this situation, numerous authors (see, e.g., refs. 11, 12, 15–18) found that fluctuations scale like $L^{2/3}$ with the size of the system, as opposed to the $L^{1/2}$ behavior in the ordered case. The problem of the interface stability in higher dimensions was adressed by a number of authors.[4,12,19] It transpired that rigidity of the interface should be expected in dimension $D=4$ and above, but there have also been arguments in favor of a critical dimension $D=5$.[12] The reasoning presented in ref. 4 invokes an Imry–Ma type argument[13] and a mapping to the random-field Ising model, both of which will be discussed in some detail below.

Some more recent results in this context concern fluctuations of objects of arbitrary codimension (see, e.g., ref. 10). They concern predictions on scaling exponents based on results obtained using the "functional renormalization group" approach of Fisher.[8] In the case of directed polymers, some interesting rigorous results were obtained by Imbrie and Spencer.[14] We will not go into the details of these developments.

Returning to the central problem of the present work, let us briefly recall the arguments for stability in four dimensions presented in ref. 4. The main observation was that an interface model of the type described above can be represented as a contour model. Let Γ be a collection of oriented loops γ_i such that either

(i) $\operatorname{int}(\gamma_i)\cap\operatorname{int}(\gamma_j)=\varnothing$

or

(ii) $\operatorname{int}(\gamma_i)\subset\operatorname{int}(\gamma_j)$, or $\operatorname{int}(\gamma_j)\subset\operatorname{int}(\gamma_i)$

The orientation σ_γ of a loop indicates whether it represents a step "up" or "down." The energy of the corresponding surface is then given in terms of Γ by

$$E(\Gamma)=\sum_i |\gamma_i| + \sum_x J_x(H_x(\Gamma)) \tag{1.2}$$

where $H_x(\Gamma)$ is naturally the height of the surface at site x, i.e.,

$$H_x(\Gamma)=\sum_{i:x\in\operatorname{int}(\gamma_i)} \sigma_{\gamma_i} \tag{1.3}$$

In ref. 4 it was argued that the Hamiltonian (1.2) strongly resembles an Ising model in a random magnetic field in dimension $d=D-1$. For the latter model has by now been proven rigorously[1,5] that the lower critical dimension, i.e., the dimension above which a phase transition takes place, equals two, which then suggests that in the interface model the critical dimension should be three. The same result is obtained by a simple Imry–Ma type argument[13]: Suppose we want to form a large contour γ. Obviously, there is a price in energy to be paid for building the wall, which equals $|\gamma|$. This will suppress contours in dimension above two, unless we gain some energy from the fact that the location of the ceiling has been changed. The amount of energy that we may expect to gain is of the order of the fluctuations of the random variable $\sum_{x \in \mathrm{int}(\gamma)} J_x(H)$, which by the central limit theorem will be of the order $|\mathrm{int}(\gamma)|^{1/2}$. For a contour of sufficiently regular shape and linear dimension L, this is of order $L^{d/2}$, while the surface energy is of order L^{d-1}. Thus, for $d>2$, the bulk term is negligible compared to the surface term, and thus no large contours will form, while in $d=2$ the two terms are comparable, and large deviations of the bulk energy will in fact delocalize the interface.

To make the argument above rigorous, it seems natural to adopt the method used by Bricmont and Kupiainen[5] in the random field model to the present situation. In the present paper we undertake the first step of this endeavor by proving the rigidity of the interface in $d>2$ dimensions for a hierarchical version of the above model. While this is putting aside the geometrical complexities associated with the renormalization of contour models (cluster expansions, etc.), it will allow us to elucidate the probabilistic aspects of the renormalization of the random variables $J_x(H)$. In fact, it is this probabilistic part in which the main differences between the random field and the interface model arise: in the RFI model, we have to deal with the renormalization of just one random variable, the magnetic field, for each site x, while in the interface model, with each site there is associated an infinite family of random variables $J_x(H)$, which, even if initially independent, will rapidly enter into interaction with each other. The extension of these results to the full model described in (1.2), (1.3) is under way.[2]

Let us describe the hierarchical model we will study. We consider a d-dimensional square of side length L^N; we divide it into L^d blocks of side length L^{n-1}, each of which is subdivided again into L^d blocks of side length L^{N-2}, and so on, until we arive at blocks of side length one. We refer to the blocks of side length L^n as the nth hierarchy. The blocks of the nth hierarchy will be labeled by the set

$$Y_n = \{ y \mid y_i = -L^{N-n}+1, ..., L^{N-n}-1 \} \tag{1.4}$$

where y_i denotes the ith component of the vector y. We denote by L^{-1} the map from Y_n to Y_{n+1} such that $L^{-1}y_i = \mathrm{Int}(y_i/L)$, i.e., the map that associates with $y \in Y_n$ its block in the next hierarchy. We also denote, for $y \in Y_{n+1}$, by Ly the collection of sites $x \in Y_n$ such that $L^{-1}x = y$.

Our surfaces will now be obtained by constructing towers above each block of each hierarchy. We denote by h_y^n, for $y \in Y_n$, the height of the tower above the block y in the nth hirarchy. Obviously the height of the surface so obtained at site x, H_x^N, is then given by the sum of the heights of the towers that contain x, i.e.,

$$H_x^N = \sum_{n=0}^{N} h_{L^{-n}x}^{(n)} \tag{1.5}$$

We will, for technical reasons, restrict the heights to take values between $-l$ and l and we will denote by Λ this set of allowed values. Our final bounds will be uniform in l and thus allow us to take l to infinity at the end.

The energy of a surface obtained in this manner will be given by

$$E(\{h\}) = \sum_{n=0}^{N} \sum_{y \in Y_n} |h_y^{(n)}| \, L^{(d-1)n} + \sum_{x \in Y_0} J_x(H_x^N) \tag{1.6}$$

The partition function $Z_N(\beta, J)$ is then given by

$$Z_N(\beta, J) = \sum_{\{h_y^{(N)}\},\ldots,\{h_y^{(0)}\}} e^{-\beta E(\{h\})} \tag{1.7}$$

The corresponding finite-volume Gibbs measures will be denoted by $\mu_{N,\beta,J}$. The mean height of the interface in the thermodynamic limit $m(\beta, J)$ is given by

$$m(\beta, J) = \lim_{N \to \infty} \mu_{N,\beta,J}(H_0^N) \tag{1.8}$$

We will assume that the $J_x(H)$ are random variables such that:

(i) All $J_x(H)$ are equally distributed.

(ii) $J_x(H)$ and $J_{x'}(H')$ are independent, if $x \neq x'$.

(iii) $\mathbb{P}(|J_x(H)| > \delta) \leqslant e^{-\delta^2/2\varepsilon^2}$, for all $\delta \geqslant \varepsilon$.

Remark 1. Note that we do not assume that the $J_x(H)$ are independent for different values of H. As we will see, such an assumption would be of no advantage for our proof, since in the renormalization process an initial independence will be destroyed rather quickly. As an aside for readers familiar with the functional renormalization group approach,[8] we may note that the fixpoints found by this method also do not correspond to independent random variables. From the point of view of applications, it is also advantageous to be able to avoid such an assumption, which may not correspond to the physical reality (one may think, for instance, of the problem of fluctuations of Peierls contours in a spin glass[3]).

We may now announce the main result of this paper:

Theorem 1.1. Let $d>2$, and assume that $J_x(H)$ is given as described above. Then there exist L_0, β_0, and ε_0 finite, such that for all $L\geqslant L_0$, $\beta\geqslant\beta_0$, and $\varepsilon\leqslant\varepsilon_0$,

$$\mathbb{P}(|m(\beta, J)|>\delta)\leqslant e^{-\delta^2/2\tilde{\varepsilon}^2} \tag{1.9}$$

for all $\delta\geqslant\varepsilon$, where $\tilde{\varepsilon}$ is of the order of ε.

Remark 2. We expect the theorem to hold with $L_0=2$; however, for technical reasons we have to choose L somewhat large.

Remark 3. The theorem implies that the interface is stable in the sense that we know, at a given site x, where to find it (namely at height zero!). It does not imply, however, that in the thermodynamic limit the height of the interface is everywhere bounded! On the contrary, we must expect that on a typical configuration towers of arbitrary height will occur "somewhere." However, the average distances between towers of height h will be of the order $e^{h^2/2(d-1)\varepsilon^2}$; furthermore, the highest towers will most likely be very thin, i.e., belong to the zeroth hierarchy, and towers of the nth hierarchy of height h will be separated by distances $e^{h^2/2(d-1)\varepsilon_n^2}$, where ε_n decreases exponentially with n.

Remark 4. In $d=2$ the interface is expected to become unstable, since the variances of the random variables $J_x^{(n)}(H)$ do not converge to zero. A detailed analysis of this situation will be presented elsewhere.[6]

The remainder of this paper is organized as follows. In Section 2 we derive the renormalization group equations for our model and the resulting formulas for $m(\beta, J)$. In Section 3 we control the flow of the renormalization group in the limit $\beta=\infty$ and prove the corresponding special case of Theorem 1.1. In Section 4 we complete the proof of the theorem for β finite.

2. THE RENORMALIZATION GROUP TRANSFORMATION

The structure of the hierarchical model invites and facilitates the evaluation of the partition function (2.7) by a successive summation over the different hierarchies of towers. That is,

$$\begin{aligned} Z_N(\beta, J) &= \sum_{\{h_y^{(N)}\}}\cdots\sum_{\{h_y^{(1)}\}} \exp\left\{-\beta\sum_{n=1}^{N}\sum_{y_n}|h_{y_n}^{(n)}|\,L^{(d-1)n}\right\} \\ &\quad\times\sum_{\{h_x^{(0)}\}}\exp\left\{-\beta\left[\sum_x |h_x^{(0)}|+J_x(H_x^N)\right]\right\} \\ &= \sum_{\{h_y^{(N)}\}}\cdots\sum_{\{h_y^{(1)}\}} \exp\left\{-\beta\sum_{n=1}^{N}\sum_{y_n}|h_{y_n}^{(n)}|\,L^{(d-1)n}\right\} \\ &\quad\times\prod_{y_1}\left(\sum_{h_x^{(0)}:\,x\in Ly_1}\exp\left\{-\beta\left[\sum_{x\in Ly_1}|h_x^{(0)}|+J_x(\tilde{H}_{y_1}+h_x^{(0)})\right]\right\}\right) \end{aligned} \tag{2.1}$$

where $\tilde{H}_y = \sum_{n=1}^{N} h_y^{(n)}$. Thus,

$$Z_N(\beta, J) = \sum_{h_y^{(N-1)}} \cdots \sum_{h_y^{(0)}} \exp\left\{-\beta L^{(d-1)}\left[\sum_{n=0}^{N-1} |h_{y_n}^{(n)}|\, L^{(d-1)n} + \sum_{y} \tilde{J}_y(H_y^{N-1})\right]\right\}$$

$$= Z_{N-1}(\beta^{(1)}, \tilde{J}) \tag{2.2}$$

where

$$\tilde{J}_y(H) = \frac{-1}{\beta L^{d-1}} \sum_{x \in Ly} \ln\left[\sum_{h_x} \exp\{-\beta[|h_x| + J_x(H+h_x)]\}\right] \tag{2.3}$$

and

$$\beta^{(1)} = \beta L^{d-1} \tag{2.4}$$

The following lemma assures that our procedure will make sense:

Lemma 2.1. Suppose the $J_x(h)$ are identically distributed random variables, independent for different values of x (but not necessarily for different values of h!!) that satisfy

(a) $\mathbb{E}(J_x(h)) = 0$

(b) $\mathbb{P}(|J_x(h)| > \delta) < e^{-\delta^2/2\varepsilon^2}$, for all δ large enough

Then the sum

$$\sum_{h=-\infty}^{\infty} e^{-\beta[|h| + J_x(H+h)]}$$

converges almost surely, and the $\tilde{J}_y(H)$ are well-defined, almost surely bounded, random variables with identical distributions, independent for different y.

Proof. Let us show first that the sums converge. Obviously, they may diverge only if, for infinitely many values of h,

$$e^{\beta[|h| + J_x(H+h)]} > e^{-\beta|h|/2}$$

On the other hand, we have by assumption that

$$\sum_{h=-\infty}^{\infty} \mathbb{P}(J_x(H+h) < -|h|/2) < \sum_{h=-\infty}^{\infty} e^{-|h|^2/8\varepsilon^2} < \infty \tag{2.5}$$

and therefore, by the Borel–Cantelli lemma,

$$\mathbb{P}((J_x(H+h) < -|h|/2 \text{ i.o.}) = 0 \quad \text{a.s.} \tag{2.6}$$

Moreover, the sum is almost surely strictly positive, its logarithm thus defined, and $\tilde{J}_y(H)$, being a finite sum of such terms, is thus a well-defined random variable. The fact that its distribution is independent of H follows immediately from the inductive assumption. ∎

The fact that the $\tilde{J}$ are identically distributed will allow us to remove their common mean and thus obtain new variables

$$J_y^{(1)}(H) \equiv \tilde{J}_y(H) - \mathbb{E}\tilde{J}_0(0) \tag{2.7}$$

which have all the properties of the original $J_x(h)$, except of course that we have not yet proven the exponential bounds (b) for its distribution, the derivation of which will constitute our main task. In terms of these new variables, the recursion (2.2) takes now the form

$$Z_N(\beta, J) = \exp\{L^{d(N-1)}\mathbb{E}J_0(0)\}\, Z_{N-1}(\beta^{(1)}, J^{(1)}) \tag{2.8}$$

This process can now be iterated to yield the following general set of recursive equations:

$$Z_N(\beta, J) = \exp\left[\sum_{k=1}^{n} L^{d(N-k)}\mathbb{E}J_0^{(k-1)}(0)\right] Z_{N-n}(\beta^{(n)}, J^{(n)}) \tag{2.9}$$

$$\beta^{(n)} = L^{(d-1)n}\beta \tag{2.10}$$

$$\begin{aligned} J_y^{(n)}(H) = {} & \frac{-1}{\beta^{(n)}} \sum_{x \in Ly} \ln\left[\sum_{h_x} \exp\{-\beta^{(n-1)}[|h_x| + J_x^{(n-1)}(H+h_x)]\}\right] \\ & + \mathbb{E}\left(\frac{1}{\beta^{(n)}} \sum_{x \in Ly} \ln\left[\sum_{h_x} \exp\{-\beta^{(n-1)}[|h_x| + J_x^{(n-1)}(H+h_x)]\}\right]\right) \end{aligned} \tag{2.11}$$

We may use these recursion relations to obtain a formula for the mean height. We have

$$\begin{aligned} & \mu_{N,\beta,J}(|H_0^N|) \\ & \quad \leqslant \frac{1}{Z_N(\beta, J)} \sum_{\{h_y^{(N)}\}} \cdots \sum_{\{h_y^{(0)}\}} \\ & \quad \times \exp\left\{-\beta \sum_{n=0}^{N} \sum_{y_n} |h_{y_n}^{(n)}|\, L^{(d-1)n} + J_x(H_x^N)\right\} \sum_{n=0}^{N} |h_0^{(n)}| \\ & \quad = \sum_{n=0}^{N} \frac{1}{Z_{N-n}(\beta^{(n)}, J^n)} \sum_{\{h_y^{(N)}\}} \cdots \sum_{\{h_y^{(n)}\}} \\ & \quad \times \exp\left\{-\beta^{(n)} \sum_{k=n}^{N} \sum_{y_k} |h_{y_k}^{(k)}|\, L^{(d-1)(k-n)} + \sum_{x \in Y_n} J_x(H_x^{N-n})\right\} |h_0^{(n)}| \end{aligned} \tag{2.12}$$

Here we have performed the sum over the first $n-1$ hierarchies in the term involving $h_0^{(n)}$. Performing now the summation over the nth hierarchy, we note that only the sum over $h_0^{(n)}$ differs from the previous construction. One thus verifies easily that

$$\frac{1}{Z_{N-n}(\beta^{(n)}, J^n)} \sum_{\{h_y^{(N)}\}} \cdots \sum_{\{h_y^{(n)}\}} \exp\left\{-\beta^{(n)} \sum_{k=n}^{N} \sum_{y_k} |h_{y_k}^{(k)}| \, L^{(d-1)(k-n)} + \sum_{x \in Y_n} J_x(H_x^{N-n})\right\} |h_0^{(n)}|$$
$$= \frac{1}{Z_{N-n-1}(\beta^{(n+1)}, J^{n+1})} \sum_{\{h_y^{(N)}\}} \cdots \sum_{\{h_y^{(n+1)}\}} \langle h_0^{(n)} \rangle_n \, (H_0^{N-n-1})$$
$$\times \exp\left\{-\beta^{(n+1)} \sum_{k=n+1}^{N} \sum_{y_k} |h_{y_k}^{(k)}| \, L^{(d-1)(k-n-1)} + \sum_{x \in Y_{n+1}} J_x(H_x^{N-n-1})\right\} \tag{2.13}$$

where

$$\langle h_0^{(n)} \rangle_n \, (H_0^{N-n-1}) \equiv \frac{\sum_h (\exp\{-\beta^{(n)}[|h| + J_0^{(n)}(H_0^{N-n-1} + h)]\}) \, |h|}{\sum_h \exp\{-\beta^{(n)}[|h| + J_0^{(n)}(H_0^{N-n-1} + h)]\}} \tag{2.14}$$

Notice that while $\langle h_0^{(n)} \rangle_n$ depends on H^{N-n-1}, its distribution is independent of it and depends only on the distribution of the $J_x^{(n)}(H)$.

Continuing to sum over the hierarchies as before, we obtain finally that

$$\mu_{N,\beta,J}(H_0^N) \leqslant \sum_{n=0}^{N} \langle h_0^{(n)} \rangle_N \tag{2.15}$$

where for $m > n$, $\langle h_0^{(n)} \rangle_m$ is defined recursively as

$$\langle h_0^{(n)} \rangle_m \, (H_0^{N-m-1})$$
$$= \frac{\sum_h (\exp\{-\beta^{(m)}[|h| + J_0^{(m)}(H_0^{N-m-1} + h)]\}) \langle h_0^{(n)} \rangle_{m-1} \, (H_0^{N-m-1} + h)}{\sum_h \exp\{-\beta^{(m)}[|h| + J_0^{(m)}(H_0^{N-m-1} + h)]\}} \tag{2.16}$$

This sets up our formalism. The proof of Theorem 1.1 will now consist of two steps: First we set up control of the random variables $J_x^{(n)}(H)$. We will show that, for $d > 2$, the $J_y^{(n)}(H)$ satisfy bounds of the form

$$\mathbb{P}(|J_y^{(n)}(H)| > \delta) < e^{-\delta^2/2\varepsilon_n^2}$$

where the ε_n converge to zero exponentially fast. This result will then be used together with formulas (2.14)–(2.16) to prove the theorem.

In the next section we will do this first in the limit $\beta \to \infty$, where the sums over h will be seen to be governed by just one term. The general case will be treated in Section 4.

3. THE CASE $\beta = \infty$

For β large, the sums over h appearing in the recursive definitions of the variables $J_y^{(n)}(H)$ are clearly dominated by the terms for which $-[|h| + J_x(H+h)]$ takes on its maximal value, and in the limit $\beta = \infty$ this becomes an exact relation, as the following lemma shows.

Lemma 3.1. Under the assumptions of Lemma 2.1,

$$\lim_{\beta \to \infty} \frac{-1}{\beta L^{d-1}} \sum_{x \in Ly} \ln \left[\sum_{h_x} \exp\{ -\beta [|h_x| + J_x(H+h_x)] \} \right]$$
$$= \frac{1}{L^{d-1}} \sum_{x \in Ly} \inf_h [|h_x| + J_x(H+h_x)] \quad \text{a.s.} \tag{3.1}$$

The proof of this lemma is left as an exercise.

Lemma 3.1 invites us to study the following simplified system of recursions:

$$J_y^{(n)}(H) = \frac{1}{L^{d-1}} \sum_{x \in Ly} \{ \inf_h [|h| + J_x^{(n-1)}(H+h)]$$
$$- \mathbb{E} \inf_h [|h| + J_x^{(n-1)}(H+h)] \} \tag{3.2}$$

Remark. We present the analysis of this particular limit for two reasons: First, the estimates are considerably easier and more transparent. Second, we want to emphasize the point that all relevant contributions that might destabilize the interface are already present here, and that finite-temperature effects only produce negligible corrections.

Proposition 3.1. Let $J_x(H)$ satisfy the assumptions introduced in Section 1. Then, if L is sufficiently large and ε small enough, the random variables $J_x^{(n)}(H)$ defined through (3.2) are identically distributed random variables, independent for different x, $\mathbb{E}J_x^{(n)}(H) = 0$, and

$$\begin{aligned} \mathbb{P}(J_x^{(n)}(H) > \delta) &\leq e^{-\delta^2/2\varepsilon_n^2} \\ \mathbb{P}(J_x^{(n)}(H) < -\delta) &\leq e^{-\delta^2/2\varepsilon_n^2} \end{aligned} \tag{3.3}$$

for all $\delta > \varepsilon_n$, and

$$\varepsilon_n^2 = c^n \varepsilon^2 \tag{3.4}$$

with $c < 1$ a constant depending on L and d.

Proof. The proof of the proposition proceeds by induction over n. We will thus assume (3.3) to hold for $n-1$ and use this to prove them for $J_y^{(n)}(H)$. To do so, we study first the variables

$$I_x(H) = \inf_h \left[|h| + J_x^{(n-1)}(H+h) \right] \tag{3.5}$$

Clearly, the $I_x(H)$ are all identically distributed and independent for different x. Moreover, it is obvious that for $\delta > 0$,

$$\mathbb{P}(I_x(H) > \delta) \leqslant \mathbb{P}(J_x^{(n-1)}(H) > \delta) \tag{3.6}$$

simply since the inf is certainly not bigger than the particular term with $h = 0$.

The more difficult part is thus to estimate $\mathbb{P}(I_x(H) < -\delta)$. However, since, by assumption, $J_x^{(n-1)}(H)$ has a very small probability to take on a large negative value, we may expect that the inf will typically occur for $h = 0$, which would give a bound like (3.6), while all other possibilities together just give a contribution of similar size. To make this idea precise, we write

$$\begin{aligned}
&\mathbb{P}(I_x(H) < -\delta) \\
&\quad = \sum_{n_h = -\infty}^{\infty} \\
&\qquad \times \mathbb{P}\left(\inf_h \left[|h| + J_x^{(n-1)}(H+h) \right] < -\delta \,\middle|\, \forall_h J_x^{(n-1)}(H+h) \in [n_h - 1, n_h]\right) \\
&\qquad \times \mathbb{P}(\forall_h J_x^{(n-1)}(H+h) \in [n_h - 1, n_h])
\end{aligned} \tag{3.7}$$

Working with the conditional probabilities gives us a better control over the infimum; in particular, we may arrange the summation over the n_h in such a way as to make explicit the minimal value that $|h| + n_h$ takes on:

$$\begin{aligned}
&\mathbb{P}(I_x(H) < -\delta) \\
&\quad = \sum_{m = -\infty}^{\infty} \sum_{n_h :\, |h| + n_h \leqslant -m} \\
&\qquad \times \mathbb{P}\left(\inf_h \left[|h| + J_x^{(n-1)}(H+h) \right] < -\delta \,\middle|\, \forall_h J_x^{(n-1)}(H+h) \in [n_h - 1, n_h]\right) \\
&\qquad \times \mathbb{P}(\forall_h J_x^{(n-1)}(H+h) \in [n_h - 1, n_h])
\end{aligned} \tag{3.8}$$

It is understood that in the sum over the n_h, equality has to hold at least for one h. Note further that, given the conditions, the inf is necessarily taken among exactly those h for which $|h|+n_h=-m$. It is thus useful to make the set X of those h for which this equality holds explicit. This yields

$$\begin{aligned}
&\mathbb{P}(I_x(H)<-\delta)\\
&\quad=\sum_{m=-\infty}^{\infty}\sum_{X}\sum_{\substack{n_h:\,|h|+n_h=-m,\,h\in X\\ |h|+n_h>-m,\,h\notin X}}\\
&\qquad\times\mathbb{P}(\inf_{h\in X}[|h|+J_x^{(n-1)}(H+h)]<-\delta\mid\forall_h J_x^{(n-1)}(H+h)\in[n_h-1,n_h])\\
&\qquad\times\mathbb{P}(\forall_h J_x^{(n-1)}(H+h)\in[n_h-1,n_h])
\end{aligned}\tag{3.9}$$

Notice that by now the event under consideration depends only on the variables $J_x^{(n-1)}(H+h)$ with $h\in X$; we may thus sum over all the other n_h, and, neglecting the restrictions on their range, get the upper bound

$$\begin{aligned}
&\mathbb{P}(I_x(H)<-\delta)\\
&\quad\leqslant\sum_{m=-\infty}^{\infty}\sum_{X}\\
&\qquad\times\mathbb{P}(\inf_{h\in X}[|h|+J_x^{(n-1)}(H+h)]<-\delta\mid\forall_{h\in X} J_x^{(n-1)}(H+h)\\
&\qquad\in[-m-|h|-1,-m-|h|])\\
&\qquad\times\mathbb{P}(\forall_{h\in X} J_x^{(n-1)}(H+h)\in[-m-|h|-1,-m-|h|])
\end{aligned}\tag{3.10}$$

In the summation over all subsets X we can fix the maximal value of $|h|$. Then,

$$\begin{aligned}
&\mathbb{P}(I_x(H)<-\delta)\\
&\quad\leqslant\sum_{m=-\infty}^{\infty}\sum_{R=0}^{\infty}\sum_{X:\,\max_{h\in X}|h|=R}\\
&\qquad\times\mathbb{P}(\inf_{h\in X}[|h|+J_x^{(n-1)}(H+h)]<-\delta\mid\forall_{h\in X} J_x^{(n-1)}(H+h)\\
&\qquad\in[-m-|h|-1,-m-|h|])\\
&\qquad\times\mathbb{P}(\forall_{h\in X} J_x^{(n-1)}(H+h)\in[-m-|h|-1,-m-|h|])
\end{aligned}$$

$$= \sum_{m=-\infty}^{\infty} \mathbb{P}(J_x^{(n-1)}(H) < -\delta \mid \forall_{h \in X} J_x^{(n-1)}(H) \in [-m-1, -m])$$

$$\times \mathbb{P}(J_x^{(n-1)}(H) \in [-m-1, -m])$$

$$+ \sum_{m=-\infty} \sum_{R=1}^{\infty} \sum_{X.\, \max_{h \in X} |h| = R}$$

$$\times \mathbb{P}(\inf_{h \in X} [|h| + J_x^{(n-1)}(H+h)] < -\delta \mid \forall_{h \in X} J_x^{(n-1)}(H+h)$$

$$\in [-m-|h|-1, -m-|h|])$$

$$\times \mathbb{P}(\forall_{h \in X} J_x^{(n-1)}(H+h) \in [-m-|h|-1, -m-|h|]) \qquad (3.11)$$

The first term is simply $\mathbb{P}(J_x^{(n-1)}(H) < -\delta)$; to estimate the second, notice that

$$\mathbb{P}(\inf_{h \in X} [|h| + J_x^{(n-1)}(H+h)] < -\delta \mid \forall_{h \in X} J_x^{(n-1)}(H+h)$$

$$\in [-m-|h|-1, -m-|h|])$$

$$= \begin{cases} 0 & \text{if } m+1 < \delta \\ \leqslant 1 & \text{if } m+1 \geqslant \delta \end{cases} \qquad (3.12)$$

and that

$$\mathbb{P}(\forall_{h \in X} J_x^{(n-1)}(H+h) \in [-m-|h|-1, -m-|h|])$$

$$\leqslant \mathbb{P}(J_x^{(n-1)}(H+R) \in [-m-R-1, -m-R])$$

$$< \exp\left[-\frac{(m+R)^2}{2\varepsilon_{n-1}^2}\right] \qquad (3.13)$$

Together with the obvious fact that the sum over the subsets of heights such that the modulus of the maximal height equals R extends over no more than 4^R terms, this allows us to get the final bound

$$\mathbb{P}(I_x(H) < -\delta) \leqslant \exp\left(-\frac{\delta^2}{2\varepsilon_{n-1}^2}\right) + \sum_{m \geqslant \delta - 1} \sum_{R=1}^{\infty} 4^R \exp\left[-\frac{(m+R)^2}{2\varepsilon_{n-1}^2}\right]$$

$$\leqslant C' \exp\left(-\frac{\delta^2}{2\varepsilon_{n-1}^2}\right) \leqslant \exp\left(-\frac{\delta^2}{2\tilde{\varepsilon}_{n-1}^2}\right) \qquad (3.14)$$

where C' is a constant that can be choosen, e.g., to equal 6, provided ε_{n-1} is sufficiently large, and $\delta \geqslant c$, and

$$\tilde{\varepsilon}_{n-1}^2 = \varepsilon_{n-1}^2 \left[1 + O\left(\frac{\varepsilon_{n-1}^2}{c^2}\right)\right]$$

Notice that the bounds thus obtained for $I_x(H)$ imply that its mean is exponentially small, and therefore virtually the same bounds hold for

$$\tilde{I}_x(H) \equiv I_x(H) - \mathbb{E}I_x(0)$$

Thus, we are almost done: the $J_y^{(n)}(H)$ are sums of the (independent!) random variables with mean zero satisfying the exponential bounds (3.14), and what is left is to extract the corresponding bounds for the sums. To do so, the following lemma is useful:

Lemma 3.2. Let X be a random variable such that

(i) $\mathbb{E}X = 0$

(ii) $\mathbb{P}(X > \delta) \leqslant e^{-\delta^2/2\varepsilon^2}$ and $\mathbb{P}(X < -\delta) \leqslant e^{-\delta^2/2\varepsilon^2}$.

Then, the Laplace transform $\mathbb{E}e^{tX}$, satisfies, for $t > 0$, the bound

$$\mathbb{E}e^{tX} \leqslant e^{c\varepsilon^2 t^2} \tag{3.15}$$

where c is a universal numerical constant (independent of t and ε) order unity.

Proof. To prove the lemma, we use the fact that

$$e^x \leqslant \begin{cases} 1 + x + \dfrac{x^2}{2} & \text{if} \quad x \leqslant 0 \\[2ex] 1 + x + \dfrac{x^2}{2} e^x & \text{if} \quad x \geqslant 0 \end{cases} \tag{3.16}$$

Let first $t \leqslant \varepsilon$. Then, by (3.16),

$$\begin{aligned} \mathbb{E}e^{tX} &= 1 + \tfrac{1}{2}\mathbb{E}(t^2 X^2 \chi_{X \leqslant 0}) + \tfrac{1}{2}\mathbb{E}(t^2 X^2 e^{tX} \chi_{X \geqslant 0}) \\ &\leqslant \exp[\tfrac{1}{2} t^2 (\mathbb{E}t^2 X^2 \chi_{X \leqslant 0} + \mathbb{E}t^2 X^2 e^{tX} \chi_{X \geqslant 0})] \end{aligned} \tag{3.17}$$

We just need to estimate the two expectations in the exponential. Note that

$$\begin{aligned} \mathbb{E}X^2 \chi_{X \leqslant 0} &= \sum_{n=0}^{\infty} \mathbb{E}(X^2 \mid -\varepsilon(n+1) < X \leqslant -\varepsilon n)\, \mathbb{P}(-\varepsilon(n+1) < X \leqslant -\varepsilon n) \\ &\leqslant \varepsilon^2 \sum_{n=0}^{\infty} (n+1)^2\, e^{-n^2/2} = C_1 \varepsilon^2 \end{aligned} \tag{3.18}$$

and similarly

$$\mathbb{E}X^2 e^{tX}\chi_{X\geqslant 0}=\sum_{n=0}^{\infty}\mathbb{E}(X^2e^{tX}\,|\,\varepsilon n\leqslant X<\varepsilon(n+1))\,\mathbb{P}(\varepsilon n\leqslant X<\varepsilon(n+1))$$

$$\leqslant \varepsilon^2\sum_{n=0}^{\infty}(n+1)^2\,e^{(n+1)-n^2/2}=C_2\varepsilon^2 \tag{3.19}$$

where the last inequality made use of the assumption $t\varepsilon\leqslant 1$.

For larger t, we must proceed in a slightly different way:

$$\mathbb{E}e^{tX}=\sum_{n=-\infty}^{\infty}\mathbb{E}(e^{tX}\,|\,\varepsilon(n-1)\leqslant X<\varepsilon n)\,\mathbb{P}(\varepsilon(n-1)\leqslant X<\varepsilon n)$$

$$=\sum_{n=0}^{\infty}\mathbb{E}(e^{tX}\,|\,-\varepsilon(n+1)\leqslant X<-\varepsilon n)\,\mathbb{P}(-\varepsilon(n+1)\leqslant X<-\varepsilon n)$$

$$+\sum_{n=0}^{\infty}\mathbb{E}(e^{tX}\,|\,\varepsilon n\leqslant X<\varepsilon(n+1))\,\mathbb{P}(\varepsilon n\leqslant X<\varepsilon(n+1))$$

$$\leqslant\sum_{n=0}^{\infty}(e^{-nt\varepsilon-n^2/2}+e^{(n+1)t\varepsilon-n^2/2}) \tag{3.20}$$

Taking into account the fact that now $t\varepsilon\geqslant 1$, the latter sum is easily bounded by

$$\mathbb{E}e^{tX}\leqslant e^{C_3t^2\varepsilon^2} \tag{3.21}$$

Combining these results and selecting the worst constant as c, we arrive at (3.15) and have proven the lemma. ∎

With the above bound on the Laplace transform of $\tilde{I}$, we now get immediately a bound on the Laplace transform of $J_y^{(n)}(H)$:

$$\mathbb{E}\exp[tJ_y^{(n)}(H)]=\mathbb{E}\left(\exp\left[\frac{t}{L^{d-1}}\sum_{x\in Ly}\tilde{I}_x(H)\right]\right)$$

$$=\prod_{x\in Ly}\mathbb{E}\left(\exp\left[\frac{t}{L^{d-1}}\tilde{I}_x(H)\right]\right)$$

$$\leqslant \varepsilon^{ct^2\varepsilon_{n-1}^2L^{2-d}}\equiv e^{\varepsilon_n^2t^2/2} \tag{3.22}$$

with $\varepsilon_n^2=\varepsilon_{n-1}^2cL^{2-d}$. Clearly, from (3.22) the exponential bounds (3.4) follow with $n-1$ replaced by n. Notice that ε_n is smaller than ε_{n-1}, provided only $d>2$ and L is chosen big enough such that $cL^{2-d}<1$. This proves the proposition. ∎

We are now ready to prove Theorem 2.1 for the particular case $\beta = \infty$. To do this, we need to control $\mathbb{P}(\langle h_0^{(n)} \rangle_N)$. Notice that, by the same reasoning used in the estimations for $J_x^{(n)}(H)$, we have

$$\begin{aligned}
&\mathbb{P}(\langle h_0^{(n)} \rangle_i (H) > \delta) \\
&\leqslant \sum_{m=0}^{\infty} \sum_X \sum_{n_h : |h| + n_h = -m} \\
&\quad \times \mathbb{P}\left(\lim_{\beta \to \infty} \frac{\sum_{h \in X} (\exp\{-\beta^{(i)}[|h| + J_0^{(i)}(H+h)]\}) \langle h_0^{(n)} \rangle_{i-1} (H+h)}{\sum_{h \in X} \exp\{-\beta^{(i)}[|h| + J_0^{(i)}(H+h)]\}} > \delta \right. \\
&\qquad \left| \forall_{h \in X} - m - \frac{1}{2} < |h| + J_0^{(i)}(H+h) \leqslant -m + \frac{1}{2} \right) \\
&\quad \times \mathbb{P}\left(\forall_{h \in X} - m - \frac{1}{2} < |h| + J_0^{(i)}(H+h) \leqslant -m + \frac{1}{2} \right) \\
&\quad + \sum_{m=-\infty}^{-1} \sum_X \sum_{n_h : |h| + n_h = -m} \\
&\quad \times \mathbb{P}\left(\lim_{\beta \to \infty} \frac{\sum_{h \in X} (\exp\{-\beta^{(i)}[|h| + J_0^{(i)}(H+h)]\}) \langle h_0^{(n)} \rangle_{i-1} (H+h)}{\sum_{h \in X} \exp\{-\beta^{(i)}[|h| + J_0^{(i)}(H+h)]\}} > \delta \right. \\
&\qquad \left| \forall_{h \in X} - m - \frac{1}{2} < |h| + J_0^{(i)}(H+h) \leqslant -m + \frac{1}{2} ; J_0^{(i)}(H) \geqslant -m - \frac{1}{2} \right) \\
&\quad \times \mathbb{P}\left(\forall_{h \in X} - m - \frac{1}{2} < |h| + J_0^{(i)}(H+h) \leqslant -m + \frac{1}{2} ; J_0^{(i)}(H) \geqslant -m - \frac{1}{2} \right) \\
&\leqslant \mathbb{P}(\langle h_0^{(n)} \rangle_{i-1} (H) > \delta) \\
&\quad + \sum_{m=0}^{\infty} \sum_{R=1}^{\infty} 4^R \mathbb{P}(\max_{|h| \leqslant R} \langle h_0^{(n)} \rangle_{i-1} (H+h) > \delta) \exp\left[-\frac{(R+m-1/2)^2}{2\varepsilon_i^2} \right] \\
&\quad + \sum_{m=-\infty}^{-1} \sum_{R=0}^{\infty} 4^R \mathbb{P}(\max_{||h| + m| \leqslant R} \langle h_0^{(n)} \rangle_{i-1} (H+h) > \delta) \\
&\quad \times \max\left(\exp\left(-\frac{R^2}{2\varepsilon_i^2} \right), \exp\left[-\frac{(m+1/2)^2}{2\varepsilon_i^2} \right] \right)
\end{aligned} \tag{3.23}$$

Notice that in the sum over the negative m we have retained the condition that the minimum was not attained for $h = 0$. Bounding the probability of the maximum by the sum over the probabilities, we arrive at the bound

$$\mathbb{P}(\langle h_0^{(n)} \rangle_i (H) > \delta) \leqslant \mathbb{P}(\langle h_0^{(n)} \rangle_{i-1} (H) > \delta)(1 + c e^{-1/8\varepsilon_i^2}) \tag{3.24}$$

Iterating this bound yields clearly

$$\mathbb{P}(\langle h_0^{(n)}\rangle_N > \delta) \leqslant \mathbb{P}(\langle h_0^{(n)}\rangle_n (0) > \delta)(1 + c' e^{-1/8\varepsilon_n^2}) \tag{3.25}$$

so that we are left with bounding $\mathbb{P}(\langle h_0^{(n)}\rangle_n (0) > \delta)$. This is done exactly as above, and yields

$$\mathbb{P}(\langle h_0^{(n)}\rangle_n (0) > \delta) \leqslant c e^{-\delta^2/2\varepsilon_n^2} \tag{3.26}$$

which can be summed over n to yield the bound claimed in Theorem 1.1.

4. THE CASE β FINITE

In this section we complete the proof of our main theorem by establishing control on the renormalization group flow for β large, but finite. In view of our discussion in the foregoing section, all we need is the following lemma.

Lemma 4.1. Let $J(h)$ be a family of identically distributed random variables, satisfying the bounds

$$\mathbb{P}(J(h) > \delta) \leqslant e^{-\delta^2/2\varepsilon^2} \tag{4.1}$$

$$\mathbb{P}(J(h) < -\delta) \leqslant e^{-\delta^2/2\varepsilon^2} \tag{4.2}$$

for $\delta > \varepsilon$.

Let

$$I(H) \equiv \frac{1}{\beta} \log \sum_{h \in A} e^{-\beta[|h| + J(H+h)]} \tag{4.3}$$

Then there exist $\varepsilon_0 > 0$ and $\beta_0 < \infty$ such that for all $\varepsilon < \varepsilon_0$ and $\beta > \beta_0$,

$$\mathbb{P}(I(H) > \delta) \leqslant e^{-\delta^2/2\varepsilon^2 c} \tag{4.4}$$

$$\mathbb{P}(I(H) < -\delta) \leqslant e^{-\delta^2/2\varepsilon^2} \tag{4.5}$$

where c is a numerical constant (of order unity), independent of ε and β.

Proof. Using that

$$\frac{1}{\beta} \log \sum_{h \in A} e^{-\beta[|h| + J(H+h)]} \geqslant -J(H)$$

we get immediatly

$$\mathbb{P}\left(\frac{1}{\beta} \log \sum_{h \in A} e^{-\beta[|h| + J(H+h)]} < -\delta\right) \leqslant \mathbb{P}(-J(H) < -\delta) \leqslant e^{-\delta^2/2\varepsilon^2} \tag{4.6}$$

which proves (4.5). The more difficult part is to prove (4.4). Although the basic idea is the same as the one used in the last section, there appear some more technicalities and we will provide some notation to deal with this. First, we introduce the following partition of the probability space of the $J(h)$: For a given positive integer l, we divide the interval Λ into l disjoint subsets $X_1, X_2,..., X_l$. Assign a strictly decreasing sequence of integers $m_1 > m_2 > \cdots > m_l$, and denote by $A(X_i)$ the event

$$\{\forall_{h \in X_i} - m_i - 1 < J(H+h) + |h| \leqslant -m_i\}$$

Note that $A(X_i)$ depends also on m_i. Let, furthermore, $\hat{X}$ denote the collection $\hat{X} = (X_1,..., X_l)$ and put

$$A(\hat{X}) \equiv \bigcap_{i=1}^{l} A(X_i)$$

The events $A(\hat{X})$ provide us with the following partition of unity:

$$1 = \sum_{l=1}^{\infty} \sum_{m_1 > m_2 > \cdots > m_l} \sum_{X_1,..., X_l}^{*} \mathbb{1}_{\{A(\hat{X})\}} \tag{4.7}$$

where the star in $\sum^*_{X_1,..., X_l}$ recalls that the sum is over subsets X_i subject to the restrictions that $X_i \cap X_j = \varnothing$ and $\bigcup_{i=1}^{l} X_i = \Lambda$.

Let us further introduce

$$Z(X_i) = \sum_{h \in X_i} e^{-\beta[|h| + J(H+h)]} \tag{4.8}$$

and set

$$\phi(X_i) = \frac{1}{\beta} \log Z(X_i) \tag{4.9}$$

For any given partition we may thus write

$$\begin{aligned} I(H) &= \frac{1}{\beta} \log \sum_{i=1}^{l} Z(X_i) \\ &= \phi(X_i) + \frac{1}{\beta} \log \left(1 + \sum_{i=2}^{l} \frac{Z(X_i)}{Z(X_1)}\right) \end{aligned} \tag{4.10}$$

Using (4.7), we may now write

$$\mathbb{P}(I(H) > \delta) = \sum_{l=1}^{\infty} \sum_{m_1 > m_2 > \cdots > m_l} \sum_{X_1,..., X_l}^{*} \mathbb{P}(I(H) > \delta \mid A(\hat{X})) \, \mathbb{P}(A(\hat{X})) \tag{4.11}$$

Notice that conditioned by $A(\hat{X})$ we may expect $I(H)$ in (4.11) to be dominated by $\phi(X_1)$. In fact, on $A(\hat{X})$ we have the following bounds:

$$m_1 + \frac{1}{\beta} \log |X_1| \leqslant \phi(X_1) \leqslant 1 + m_1 + \frac{1}{\beta} \log |X_1| \tag{4.12}$$

and

$$\frac{1}{\beta} \log \left(1 + \sum_{i=2}^{l} \frac{Z(X_i)}{Z(X_1)} \right) \leqslant \frac{1}{\beta} \sum_{i=2}^{l} Z(X_i)\, e^{-m_1 \beta} \tag{4.13}$$

It will turn out to be necessary to treat separately the parts of the sum (4.11) in which $m_1 \geqslant 0$ and $m_1 < 0$, respectively. Put

$$S^+ = \sum_{l=1}^{\infty} \sum_{m_1 > m_2 > \cdots > m_l}^{+} \sum_{X_1, \ldots, X_l}^{*} \mathbb{P}(I(H) > \delta \mid A(\hat{X}))\, \mathbb{P}(A(\hat{X})) \tag{4.14}$$

and

$$S^- = \sum_{l=1}^{\infty} \sum_{m_1 > m_2 > \cdots > m_l}^{-} \sum_{X_1, \ldots, X_l}^{*} \mathbb{P}(I(H) > \delta \mid A(\hat{X}))\, \mathbb{P}(A(\hat{X})) \tag{4.15}$$

The $\pm$ superscripts on the sum over the m_i recall that it extends over positive or negative m_1, respectively.

We will consider first S^+, which naturally is expected to give the main contribution. Let us write, for some $0 < \alpha < 1$ to be chosen later (e.g., $\alpha = 1/2$ will turn out to be a possible, but maybe not optimal, choice),

$$\delta = \sum_{i=1}^{\infty} \delta \alpha^{i-1} (1 - \alpha)$$

and notice that, since $\sum_{i=1}^{l} \delta \alpha^{i-1}(1-\alpha) < \delta$,

$$\begin{aligned}
&\mathbb{P}(I(H) > \delta \mid A(\hat{X})) \\
&\quad \leqslant \mathbb{P}\left(\phi(X_1) + \frac{1}{\beta} \sum_{i=2}^{l} \frac{Z(X_i)}{Z(X_1)} > \sum_{i=1}^{l} \delta \alpha^{i-1}(1-\alpha) \,\middle|\, A(\hat{X}) \right) \\
&\quad \leqslant \mathbb{P}(\phi(X_1) > \delta(1-\alpha) \mid A(\hat{X})) \\
&\quad\quad + \sum_{i=2}^{l} \mathbb{P}\left(\frac{1}{\beta} \frac{Z(X_i)}{Z(X_1)} > \delta \alpha^{i-1}(1-\alpha) \,\middle|\, A(\hat{X}) \right)
\end{aligned} \tag{4.16}$$

We consider the first term in (4.16) separately:

$$\begin{aligned}
S_1^+ &\equiv \sum_{l=1}^{\infty} \sum_{m_1 > m_2 > \cdots > m_l}^{-} \sum_{X_1, \ldots, X_l}^{*} \mathbb{P}(\phi(X_1) > \delta(1-\alpha) \mid A(\hat{X}))\, \mathbb{P}(A(\hat{X})) \\
&\leqslant \sum_{m_1 \geqslant 0} \sum_{X_1 \subset A} \mathbb{P}(\phi(X_1) > \delta(1-\alpha) \mid A(X_1))\, \mathbb{P}(A(X_1))
\end{aligned} \tag{4.17}$$

where we have summed over those conditions upon which the event under consideration does not depend. The estimation of (4.16) is now very similar to that of Eq. (3.10) of the last section. We have

$$\begin{aligned}
S_1^+ &\equiv \sum_{m_1\geqslant 0}\sum_{R=0}^{\infty}\sum_{X_1:\max_{h\in X_1}|h|=R} \mathbb{P}(\phi(X_1)>\delta(1-\alpha)\mid A(X_1))\,\mathbb{P}(A(X_1))\\
&\leqslant \mathbb{P}(-J(H)>\delta(1-\alpha))\\
&\quad+\sum_{m_1\geqslant 0}\sum_{R=1}^{\infty}\sum_{X_1:\max_{h\in X_1}|H|=R} \mathbb{P}(\phi(X_1)>\delta(1-\alpha)\mid A(X_1))\,\mathbb{P}(A(X_1))
\end{aligned} \tag{4.18}$$

Using (4.12), we have

$$\mathbb{P}(\phi(X_1)>\delta(1-\alpha)\mid A(X_1))=0 \quad \text{if } (1+m_1)+\frac{1}{\beta}\log|X_1|\leqslant\delta(1-\alpha)$$

Therefore, as in (3.14),

$$\begin{aligned}
S_1^+ &\leqslant \mathbb{P}(-J(H)>\delta(1-\alpha))\\
&\quad+\sum_{R=1}^{\infty}\sum_{m_1>\delta(1-\alpha)-1-(1/\beta)\log(2R+1)} 4^R\exp\left[-\frac{(m_1+R)^2}{2\varepsilon^2}\right]\\
&\leqslant \exp\left[-\frac{\delta^2(1-\alpha)^2}{2\varepsilon^2}\right]+C\exp\left\{-\frac{[\delta(1-\alpha)-\log 3/\beta]^2}{2\varepsilon^2}\right\}
\end{aligned} \tag{4.19}$$

We consider now the second term on the right-hand side of (4.16),

$$\begin{aligned}
S_2^+ &\equiv \sum_{l=1}^{\infty}\sum^{+}_{m_1>m_2>\cdots>m_l}\sum^{*}_{X_1,\dots,X_l}\sum_{i=2}^{\infty}\\
&\qquad\times\mathbb{P}\left(\frac{1}{\beta}\frac{Z(X_i)}{Z(X_1)}>\delta\alpha^{i-1}(1-\alpha)\,\middle|\,A(\hat{X})\right)\mathbb{P}(A(\hat{X}))\\
&\leqslant \sum_{i=2}^{\infty}\sum_{l=i}^{\infty}\sum^{+}_{m_1>m_2>\cdots>m_l}\sum^{*}_{X_1,\dots,X_l}\\
&\qquad\times\mathbb{P}\left(\frac{1}{\beta}\frac{Z(X_i)}{Z(X_1)}>\delta\alpha^{i-1}(1-\alpha)\,\middle|\,A(\hat{X})\right)\mathbb{P}(A(\hat{X}))\\
&\leqslant \sum_{i=2}^{\infty}\sum_{m_1\geqslant 0}\sum_{m_i\leqslant m_1-i+1}\sum_{X_1\cap X_i=\varnothing}\\
&\qquad\times\mathbb{P}\left(\frac{1}{\beta}\frac{Z(X_i)}{Z(X_1)}>\delta\alpha^{i-1}(1-\alpha)\,\middle|\,A(X_1)\cap A(X_i)\right)\\
&\qquad\times\mathbb{P}(A(X_1)\cap A(X_i))
\end{aligned} \tag{4.20}$$

Using (4.14), we have on $A(X_1)\cap A(X_i)$

$$\frac{1}{\beta}\frac{Z(X_i)}{Z(X_1)}\leqslant\frac{1}{\beta}\,|X_i|\,e^{\beta(m_i+1-m_1)} \tag{4.21}$$

Therefore

$$\mathbb{P}\left(\frac{1}{\beta}\frac{Z(X_i)}{Z(X_1)}>\delta\alpha^{i-1}(1-\alpha)\,\middle|\,A(X_1)\cap A(X_i)\right)=0$$

if $(1/\beta)\,|X_i|\,e^{\beta(m_i+1-m_1)}\leqslant\delta\alpha^{i-1}(1-\alpha)$. Using this, we obtain finally

$$\begin{aligned}
S_2^+ &\leqslant \sum_{i=2}^{\infty}\sum_{m_1\geqslant 0}\sum_{m_i\leqslant m_1-i+1}\sum_{|X_i|>\beta\delta\alpha^{i-1}(1-\alpha)\exp[\beta(m_1-m_i-1)]}\mathbb{P}(A(X_i))\\
&\leqslant \sum_{i=2}^{\infty}\sum_{m_1\geqslant 0}\sum_{m_i\leqslant m_1-i+1}\sum_{R>(\beta/2)\,\delta\alpha^{i-1}(1-\alpha)\exp[\beta(m_1-m_i-1)]}\\
&\qquad\times 4^R\exp[-(m_i+R)^2/2\varepsilon^2]\\
&\leqslant C\sum_{i=2}^{\infty}\sum_{m_1\geqslant 0}\sum_{m_i\leqslant m_1-i+1}2^{\beta\delta\alpha^{i-1}(1-\alpha)\exp[\beta(m_1-m_i-1)]}\\
&\qquad\times\exp\left\{-\frac{\{m_i+(\beta/2)\,\delta\alpha^{i-1}(1-\alpha)\exp[\beta(m_1-m_i-1)]\}^2}{2\varepsilon^2}\right\}\\
&\leqslant C'\sum_{i=2}^{\infty}\sum_{m_1\geqslant 0}\sum_{\Delta\geqslant i-1}2^{\beta\delta\alpha^{i-1}(1-\alpha)\exp[\beta(\Delta-1)]}\\
&\qquad\times\exp\left\{-\frac{\{-\Delta+m_1+(\beta/2)\,\delta\alpha^{i-1}(1-\alpha)\exp[\beta(\Delta-1)]\}^2}{2\varepsilon^2}\right\}\\
&\leqslant C''\exp\left\{-\frac{[(\beta/2)\,\delta\alpha^{i-1}(1-\alpha)-1]^2}{2\varepsilon^2}\right\}
\end{aligned} \tag{4.22}$$

We now turn to the remaining term, S^- [see (4.15)]. Using (4.10) and (4.13), we get

$$\begin{aligned}
S^- &\leqslant \sum_{l=1}^{\infty}\sum_{m_1>m_2>\cdots>m_l}^{-}\sum_{X_1,\dots,X_l}^{*}\\
&\qquad\times\mathbb{P}\left(\frac{1}{\beta}\sum_{i=2}^{l}\frac{Z(X_i)}{Z(X_1)}>\delta-\frac{1}{\beta}\log|X_1|-m_1\,\middle|\,A(\hat{X})\right)\mathbb{P}(A(\hat{X}))
\end{aligned} \tag{4.23}$$

We decompose the sum $\sum^*_{X_1,\dots,X_l}$ into $\sum^*_1$ and $\sum^*_2$, where $\sum^*_1$ is over $X_1,\dots,X_l$ with the condition that $|X_1|\leqslant e^{\beta(\delta-m_1)/2}$ and $\sum^*_2$ is over $X_1,\dots,X_l$

with $|X_1| > e^{\beta(\delta - m_1)/2}$. We call the corresponding terms in (4.23) S_1^- and S_2^-, respectively.

Clearly,

$$
\begin{aligned}
S_2^- &\leqslant \sum_{m_1 < 0} \sum_{|X_1| > e^{\beta(\delta - m_1)/2}} \mathbb{P}(A(X_1)) \\
&\leqslant \sum_{m_1 < 0} \sum_{R > e^{\beta(\delta - m_1)/2}/2} 4^R e^{-(R + m_1)^2/2\varepsilon^2} \qquad (4.24)
\end{aligned}
$$

Now $R > e^{\beta(\delta - m_1)/2}/2$ implies $R + m_1 > 0$ (for β large enough) and so we get

$$
\begin{aligned}
S_2^- &\leqslant \sum_{m_1 < 0} 2^{e^{\beta(\delta - m_1)/2}} \exp\left[-\frac{(m_1 + e^{\beta(\delta - m_1)/2}/2)^2}{2\varepsilon^2} \right] \\
&\leqslant C \exp\left[-\frac{(e^{\beta(\delta + 1)/2} - 1)^2}{2\varepsilon^2} \right] \qquad (4.25)
\end{aligned}
$$

Finally, the term S_1^- is estimated along the same lines as S_2^+, the only difference being that δ is replaced by $(\delta - m_1)/2$, and m_1 is of course summed only over negative values. Looking at (4.22), we see that therefore we get

$$
\begin{aligned}
S_1^- \leqslant C \sum_{i=2}^{\infty} \sum_{m_1 < 0} \sum_{\Delta \geqslant i-1} & 4^{(\beta/4)(\delta - m_1)\alpha^{i-1}(1-\alpha)\exp[\beta(\Delta - 1)]} \\
& \times \exp\left\{ -\frac{\{-\Delta + m_1 + (\beta/4)(\delta - m_1)\,\alpha^{i-1}(1-\alpha)\exp[\beta(\Delta - 1)]\}^2}{2\varepsilon^2} \right\}
\end{aligned}
\qquad (4.26)
$$

The presence of the $-\beta m_1$ term in the exponent allows us to perform the sum over the negative m_1 without a problem. We get

$$
S_1^- \leqslant C' \exp\left\{ -\frac{[-2 + (\beta/4)(\delta + 1)\,\alpha^{i-1}(1-\alpha)]^2}{2\varepsilon^2} \right\} \qquad (4.27)
$$

Collecting the terms from (4.19), (4.22), (4.25), and (4.27), we have

$$
\begin{aligned}
\mathbb{P}(I(H) > \delta) &\leqslant C \exp\left[-\frac{(\delta(1-\alpha) - \log 3/\beta)^2}{2\varepsilon^2} \right] \\
&\leqslant \exp\left(-\frac{\delta^2}{2\varepsilon^2 c} \right) \qquad (4.28)
\end{aligned}
$$

if $\delta > \varepsilon$, for some constant c. This proves Lemma 4.1. ∎

At this point we may use Lemma 3.2 from the last section to obtain immediately the analogue of Proposition 3.1 for the case β finite. Now the proof of Theorem 1.1 proceeds in exactly the same fashion as in the case $\beta = \infty$, using the partition of unity (4.7) and making similar computations as in formulas (4.14)–(4.28). ▮

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